Radiation Field of a Uniformly Moving Charge in an Anisotropic Medium

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Fourier integrals are set up for the field of a point charge moving uniformly in an arbitrary direction in a uniaxial medium anisotropic in ε only. The integrals break up into several parts two of which yield the ordinary and extraordinary cones with uniform azimuthal potential distribution. The remaining integrals neither contribute to the energy radiated nor affect the size and the shape of the cones, but merely distort the field within the cones. The integrals are evaluated exactly in the non-dispersive case and closed expressions for the potential are obtained. In the dispersive case, the radiation field is determined by using the asymptotic form of the Hankel functions occurring in the integrand. The resulting expressions exhibit the high azimuthal asymmetry characteristic of anisotropic fields. From the expressions derived for a pure dielectric the potential in a doubly anisotropic medium is obtained, without a fresh calculation, by making appropriate substitutions for the coordinates of the field point and the components of the dielectric tensor.

1. Introduction

Apart from its intrinsic theoretical interest, the radiation field of moving charges in anisotropic media assumes practical significance, in that plasma in a magnetic field behaves like a birefringent medium [1]. Passage of fast charged particles through such a medium is accompanied by emission of o- and e-types of waves [2]. The low-frequency hiss from the exosphere [3], emission of bursts of non-thermal radiation from sunspots [4], and a possible method of accelerating charged particles using the inverse Cherenkov effect [5] are some of the problems of practical interest that may have a bearing on the present topic.

In a recent paper [6] (to be referred to as I), we investigated the field of a point charge moving uniformly in a uniaxial dielectric medium. Fourier integrals for the field were set up for motion perpendicular to the optic axis and the field was calculated in the special case of an isotropic dispersive medium on the basis of a simplified model of dispersion. We also promised to report the results of our final investigations on the problem. This we now do, removing the restrictions on the direction of motion of the charge and the dispersive properties of the medium. The Fourier integral is again found to break up into three terms, two of which are essentially isotropic integrals and give rise to the circular (o-) and elliptic (e-) cones. An application of the convolution theorem of the Fourier transform converts the remaining term into an integral over the isotropic field similar to (13) of I. This term neither contributes to the total energy radiated, nor affects the size and shape of the field cones, but merely represents a distortion of the potential distribution within the cones. Evaluating this distortion integral, we obtain an exact expression for the total potential in a non-dispersive medium.

It becomes difficult to carry out the integration when dispersion is taken into account but no simplifying assumption is made about the dispersive properties of the medium. The difficulties are partly obviated by using the asymptotic expansion of the Hankel functions occurring in the integrand, and it is found that in the radiation zone, the distortion terms too become formally identical to the isotropic integral, except for a neat factor containing the azimuthal angle φ around the axis of each cone. The o- and e-distortion integral combine now with their respective cone terms and give rise to the distinctive feature of the anisotropic field, namely its highly asymmetric azimuthal distribution. Thus, the supplementary integration arising out of the Faltung brings about a redistribution of the uniform potential of the field cones and a corresponding fanning out of the light intensity around the radiation cones.

2. The Distortion Integrals and the Non-dispersive Field

On breaking a part of it into partial fractions, the fundamental Fourier integral [(1) of I or (28) of Ref. [7]] for the scalar potential Φ of a charge moving uniformly at an angle $\Omega (0 \leqq \Omega \leqq \pi/2)$ with the

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optic axis can be written as

$$\frac{8\pi^3}{e}\Phi = \int \frac{\gamma^2}{\varepsilon_2} e^{i(k_1 x_1 + k_2 x_2)} g_e(k_2)\,dk_2\,dk_1 - \beta^2\gamma^2 \int e^{i(k_1 x_1 + k_2 x_2)} g_0(k_2)\,dk_2\,dk_1 \quad (1\,a, b)$$

$$+\frac{\beta^2\gamma^3}{2}\int \frac{k_1 e^{i(k_1 x_1 + k_2 x_2)}}{\sin\Omega\cos\Omega_0}\left[\frac{(\cos\Omega+\cos\Omega_0)^2}{k_2 - p_+ k_1} - \frac{(\cos\Omega-\cos\Omega_0)^2}{k_2 + p_- k_1}\right]\{g_e(k_2) - g_0(k_2)\}\,dk_2\,dk_1 \quad (1\,c)$$

where,

$$g_0(k_2) = \int \frac{e^{ik_3 x_3}}{k_3^2 - k_0^2}\,dk_3, \quad g_e(k_2) = \int \frac{e^{ik_3 x_3}}{k_3^2 - k_e^2}\,dk_3,$$

$$k_0^2 = a_0^2 k_1^2 - k_2^2, \quad k_e^2 = \Lambda^2(a_e^2 k_1^2 - k_2'^2) \quad \text{and} \quad k_2' = k_2 + (\gamma\,\varepsilon_{12}/\varepsilon_{22})\,k_1 .$$

As explained in I, the k_3-integration can be performed by using the residue theorem with some prescription for going round the poles lying on the real axis. The functions, thus obtained, have the Fourier transforms

$$C_0(k_1, x_2, x_3) = \int e^{ik_2 x_2} g_0(k_2)\,dk_2 = i\pi^2[\operatorname{sgn} k_1\cdot J_0\{a_0|k_1|(x_2^2+x_3^2)^{1/2}\} + i N_0\{a_0|k_1|(x_2^2+x_3^2)^{1/2}\}], \quad (2\,a)$$

$$C_e(k_1, x_2, x_3) = \exp\left\{\frac{i\gamma\,\varepsilon_{12} k_1 x_2}{\varepsilon_{22}}\right\}\int e^{ik_2 x_2} g_e(k_2)\,dk_2$$

$$= \frac{i\pi^2}{\Lambda}[\operatorname{sgn} k_1\cdot J_0\{a_e|k_1|(x_2^2+\Lambda^2 x_3^2)^{1/2}\} + i N_0\{a_e|k_1|(x_2^2+\Lambda^2 x_3^2)^{1/2}\}] . \quad (2\,b)$$

On applying the convolution theorem [6], the scalar potential is brought to the form

$$\frac{8\pi^3}{e}\Phi = \int_{-\infty}^{\infty}\frac{\gamma^2}{\varepsilon_2}\exp\{i k_1 x_{1(e)}\}\,C_e(k_1,x_2,x_3)\,dk_1 - \beta^2\gamma^2\int_{-\infty}^{\infty}\exp\{i k_1 x_{1(0)}\}\,C_0(k_1,x_2,x_3)\,dk_1 \quad (3\,a, b)$$

$$+ i\beta^2\gamma^2\sum_{\pm}\int_{-\infty}^{\infty}\frac{k_1\sin\Omega\cos\Omega_0\,\gamma_\pm^2}{2\gamma}\times$$

$$\times[\int_0^\infty \exp\{i k_1(x_{1(e)} + p_{\pm(e)}t)\}\,C_e(k_1, x_2\mp t, x_3)\,dt - \quad (3\,c)$$

$$-\int_0^\infty \exp\{i k_1(x_{1(0)} + p_{\pm(0)}t)\}\,C_0(k_1, x_2\mp t, x_3)\,dt]\,dk_1$$

where $\Sigma_\pm$ stands for summation of two terms, one with the upper sign and one with the lower sign. In these expressions,

$$\cos^2\Omega_0 = (1/\beta^2)\,\varepsilon_2, \quad \Lambda^2 = \varepsilon_{22}/\varepsilon_2, \quad a_0^2 = \gamma^2(\beta^2\varepsilon_2 - 1), \quad a_e^2 = \frac{\gamma^2(\beta^2\varepsilon_{22}-1)\,\varepsilon_1\varepsilon_2}{\varepsilon_{22}^2},$$

$$x_{1(0)} = x_1, \quad x_{1(e)} = x_1 - (\gamma\,\varepsilon_{12}/\varepsilon_{22})\,x_2, \quad p_{\pm(0)} = p_\pm, \quad p_{\pm(e)} = p_\pm \pm (\gamma\,\varepsilon_{12}/\varepsilon_{22}),$$

$$p_\pm = \frac{\gamma}{\sin\Omega\cos\Omega_0}[1 \pm m^2\cos\Omega\cos\Omega_0], \quad \gamma_\pm = m\gamma\left|\frac{\cos\Omega\pm\cos\Omega_0}{\sin\Omega\cos\Omega_0}\right|, \quad \text{and} \quad m = 1 .$$

We note the useful identities

$$[p_{\pm(0)}^2 - a_0^2] = \Lambda^2[p_{\pm(e)}^2 - a_e^2] = \gamma_\pm^2 .$$

In the special case of motion perpendicular to the optic axis,

$$p_{\pm(e)} = p_{\pm(0)} = p_\pm = \gamma\beta\sqrt{\varepsilon_2}, \quad \gamma_\pm = \gamma, \quad x_{1(e)} = x_{1(0)} = x_1,$$

and Eq. (3) reduces to the integrals given in Sect. 2 of I.

The integrals (3 a, b) give the o- and e-cones and (3 c) produces distortion of the field inside the cones. In evaluating (3 c) in the non-dispersive case we carry out the integration over t and k_1 in the reverse order. Performing the k_1-integration first we get integrals of the type

$$I = \frac{\partial}{\partial x_1} \int_0^\infty F(x, t)\, \mathrm{d}t \tag{4}$$

where $F(x, t)$ is the isotropic integral, with x_1 replaced by $(x_1 + p\,t)$ and x_2 replaced by $(x_2 - t)$. We thus have,

$$F(x,t) = 0 \quad \text{for} \quad (x_1 + p\,t) > 0 \quad \text{Reg. A}, \tag{5 a}$$

$$= 0 \quad \text{for} \quad \left.\begin{matrix}(x_1 + p\,t) < 0\\ y < 0\end{matrix}\right\} \quad \text{Reg. B}, \tag{5 b}$$

$$= \frac{-4\,i}{\sqrt{y}} \quad \text{for} \quad \left.\begin{matrix}(x_1 + p\,t) < 0\\ y > 0\end{matrix}\right\} \quad \text{Reg. C} \tag{5 c}$$

where $\quad y = (x_1 + p\,t)^2 - a^2\,[\,(x_2 - t)^2 + x_3^2]$.

Since the argument applies equally well to any one of the four integrals in (3 c), we have dropped the subscripts on $p_{\pm\binom{0}{e}}$, $a_{\binom{0}{e}}$ and $x_{1\binom{0}{e}}$ in Eq. (5) above.

The region of (5) into which the integrand of (4) falls is determined by the coordinates x_i of the field point, and the value of t under consideration. As t, p, a are all positive, the integrand is always in Reg. A and is identically zero for all field points ahead of the charge $(x_1 > 0)$. For points behind the charge, it is helpful in the further analysis to make a plot of y against t. Since $(p^2 - a^2)$ is positive in all cases, this is a parabola with its arms upwards (Figure 1). Whatever be the value of y, we are again in Reg. A for all $t > -x_1/p$. At $t = -x_1/p$, the parabola always lies below the t axis. If the integral is to be at all non-vanishing, the parabola must start above the t axis at the origin. Otherwise, the integrand starts in Reg. B and goes over into Reg. A as t increases, and the integral vanishes. But $(y)_{t=0} = 0$ is just the equation of the field cone $x_1^2 = a^2\,(x_2^2 + x_3^2)$. Hence, the distortion terms contribute only when the field point lies within the backward cone, and the integration terminates at the cross-over point $t = t_1$, the smaller positive root of the quadratic $y = 0$ in t. The integration and the subsequent differentiation of (4) can now be carried out in a straightforward manner, and the non-dispersive potential within the cones is obtained as $\Phi = \Phi_0 + \Phi_e$, where

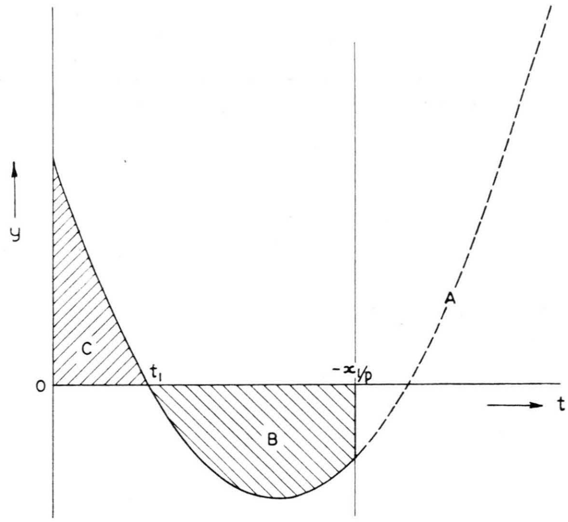

Fig. 1. The $y-t$ parabola.

$$\Phi_0 = -\frac{e\,\beta^2\,\gamma^2}{2\,\pi}\left[\frac{1}{R_0} - \frac{\sin\Omega\cos\Omega_0}{2\,\gamma}\sum_{\pm}\frac{\gamma_\pm\cdot\left[p_{\pm(0)} + \dfrac{x_{1(0)}\,\gamma_\pm}{R_0}\right]}{\gamma_\pm\cdot R_0 + \{p_{\pm(0)}\,x_{1(0)} \pm a_0^2\,x_2\}}\right], \tag{6 a}$$

$$\Phi_e = \frac{e\,\beta^2\,\gamma^2}{2\,\pi}\left[\frac{m^2\cos^2\Omega_0}{V\,R_e} - \frac{\sin\Omega\cos\Omega_0}{2\,\gamma}\sum_{\pm}\frac{\gamma_\pm\cdot\left[p_{\pm(e)} + \dfrac{x_{1(e)}\,\gamma_\pm}{\Lambda\,R_e}\right]}{\gamma_\pm\cdot\dfrac{R_e}{\Lambda} + \{p_{\pm(e)}\,x_{1(e)} \pm a_e^2\,x_2\}}\right]. \tag{6 b}$$

$$R_0^2 \equiv [x_{1(0)}^2 - a_0^2\,\varrho_0^2] = 0 \quad \text{and} \quad R_e^2 \equiv [x_{1(e)}^2 - a_e^2\,\varrho_e^2] = 0$$

are the equations of the ordinary and extraordinary cones, respectively, with $\varrho_0^2 = x_2^2 + x_3^2$ and $\varrho_e^2 = x_2^2 + (\varepsilon_{22}/\varepsilon_2)\,x_3^2$. The nature of distortion represented by the $\Sigma_\pm$ terms of (6 a, b) can now be analysed.

But these expressions diverge in the region of utmost interest, the surface of the respective cones. For removing the divergences it is necessary to consider the more realistic case of a dispersive medium.

3. Azimuthal Distribution of the Radiation Field

Since $\gamma_\pm$, $\alpha\binom{0}{e}$, $p\pm\binom{0}{e}$ are all functions of $|k_1|$ (or, the frequency), the k_1-integration cannot be performed easily when dispersion is taken into account. Let the dielectric constants ε_i have some steady values $\bar{\varepsilon}_i$ upto a frequency ω^0 above which they may be any complicated functions of the frequency. As dielectrics generally have their first absorption band in the microwave region, ε_i retain their d. c. values upto these frequencies. We can now split up each of the k_1 integrals in (3) into a low-frequency part with ε_i replaced by $\bar{\varepsilon}_i$, and a high-frequency part with $\varepsilon_i(k_1)$ determined by the dispersion curve of the medium. These can be evaluated separately and the results of integration combined. This apparently innocuous step of splitting and rejoining them, leaves the integrals (3 a, b) unaffected, but simplifies the distortion integrals (3 c) beyond recognition. It turns out that the integration over t (which can now be performed first) merely introduces a φ-dependent factor into the integrand, and the remaining k_1 integrals combine neatly with (3 a, b).

A typical high-frequency integral of (3 c) is of the form

$$\int_{k_1^0}^{\infty} \gamma^2 k_1 \cos\Omega_0\, e^{ik_1x_1} \int_0^{\infty} e^{ik_1pt} H_0^{(1)} \left[\alpha k_1 \{(x_2-t)^2+x_3^2\}^{1/2}\right] dt\, dk_1 , \tag{7 a}$$

$$+\int_{k_1^0}^{\infty} \gamma^2 k_1 \cos\Omega_0\, e^{-ik_1x_1} \int_0^{\infty} e^{-ik_1pt} H_0^{(2)} \left[\alpha k_1 \{(x_2-t)^2+x_3^2\}^{1/2}\right] dt\, dk_1 \tag{7 b}$$

in which p, α and ε_i are given functions of k_1. On changing the variable t to (x_2-t), the integrals over t in (7 a) can be written as

$$e^{ipk_1x_2} \int_{-\infty}^{x_2} e^{-ipk_1t} H_0^{(1)} \left[\alpha k_1 (t^2+x_3^2)^{1/2}\right] dt = e^{ipk_1x_2} \int_{-\infty}^{\infty} e^{-ipk_1t} H_0^{(1)} \left[\alpha k_1 (t^2+x_3^2)^{1/2}\right] dt \tag{8 a}$$

$$- e^{ipk_1x_2} \int_{x_2}^{\infty} e^{-ipk_1t} H_0^{(1)} \left[\alpha k_1 (t^2+x_3^2)^{1/2}\right] dt . \tag{8 b}$$

The integral (8 a) is in a standard form and can be evaluated exactly. However, it drops out from the final result, cancelling with a similar term coming from the other cone. In evaluating (8 b), we make use of the asymptotic form of the Hankel function. In the region of integration, $k_1 > k_1^0$, $t > x_2$ and thus the argument of H_0 is always greater than $\alpha\varrho k_1^0$. This places a lower bound on the value of ϱ above which our expressions are justified. For most dielectrics, this is of the order of 10 cm, a distance typical of the beginning of the radiation zone of the Cherenkov effect [8].

To the first order in $(1/k_1)$, 8 (b) reduces to

$$-\sqrt{\frac{2}{\pi}} \cdot \frac{\exp[i(p k_1 x_2 - \pi/4)]}{(\alpha k_1)^{1/2}} \cdot P + O(1/k_1^2) \tag{9}$$

where

$$P = \int_{x_2}^{\infty} \frac{\exp[i k_1\{-p t + \alpha(t^2+x_3^2)^{1/2}\}]}{(t^2+x_3^2)^{1/4}}\, dt . \tag{10}$$

This integral is of the form $\int u_0(t) \exp\{k_1 f(t)\}\, dt$ which can be expanded in inverse powers of k_1, by successive partial integration as follows:

$$\int u_0(t) \exp\{k_1 f(t)\}\, dt = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{k_1^{\,i}} \cdot v_i(t) \exp\{k_1 f(t)\} + \frac{(-1)^n}{k_1^{\,n}} \int u_n(t) \exp\{k_1 f(t)\}\, dt \tag{11}$$

where

$$v_i(t) = u_{i-1}(t)/f'(t) \quad \text{and} \quad u_i(t) = v_i'(t) \quad \text{for } i \geqq 1 .$$

Since the higher terms in the asymptotic expansion of the Hankel functions can be treated in a similar manner, the field can, in principle, be determined with any desired accuracy. For the present purpose, however, it suffices to retain terms only upto the first order in $(1/k_1)$, and write

$$P = \frac{1}{i k_1 \sqrt{\varrho}} \left[\frac{\exp\{i k_1(-p x_2 + a \varrho)\}}{p - a \cos\varphi} \right] + O(1/k_1^2) \tag{12}$$

where $\cos\varphi = x_2/\varrho$. Substituting (12) in (9), we notice that the resulting expression can be recast into a Hankel function asymptotically to the same order in $(1/k_1)$. (7) now takes the form

$$\int_{|k_1|>k_1^0} \gamma^2 \cos\Omega_0 \left\{ \frac{e^{ik_1x_1} C(k_1, x_2, x_3)}{(p - a\cos\varphi)} \right\} \mathrm{d}k_1 \tag{13}$$

with $C(k_1, x_2, x_3)$ given by Equation (2). The distortion integrals (13) are thus formally identical to the cone-terms of (3 a, b) apart from the factor involving the azimuthal angle φ. When these are combined with the high-frequency integrals of (3 a, b), we obtain

$$\Phi_{\text{high}} = -\frac{e\beta^2\gamma^2}{8\pi^3} \int_{|k_1|>k_1^0} F(\varepsilon, \varphi) e^{ik_1x_1} \cdot C(k_1, x_2, x_3)\, \mathrm{d}k_1, \tag{14}$$

where, the explicit form of the angular factors, $F(\varepsilon, \varphi)$, is given by Equation (19).

As one approaches the surface of a cone, the non-dispersive potential (6) tends to the expression

$$\Phi_{\text{nd}} = -\frac{e\beta^2\gamma^2}{2\pi} F(\varepsilon, \varphi) \left\{ \frac{1}{R} \right\} \tag{15}$$

with the *same* angular factor as in the high-frequency integral (14). The low-frequency integrals of Eq. (3) can be obtained as

$$\Phi_{\text{low}} = \int_{|k_1|<k_1^0} f(\bar\varepsilon, k_1)\, \mathrm{d}k_1 = \int_{-\infty}^{\infty} f(\bar\varepsilon, k_1)\, \mathrm{d}k_1 - \int_{|k_1|>k_1^0} f(\bar\varepsilon, k_1)\, \mathrm{d}k_1 . \tag{16 a, b}$$

These two integrals (16 a, b) are identical respectively to (15) and (14) except that ε in the latter is replaced by $\bar\varepsilon$. Noting that $\{1/R\}$ is the Fourier transform of $C(k_1, x_2, x_3)$, we thus have

$$\Phi_{\text{low}} = -\frac{e\beta^2\gamma^2}{8\pi^3} F(\bar\varepsilon, \varphi) \int_{|k_1|<k_1^0} e^{ik_1x_1} C(k_1, x_2, x_3)\, \mathrm{d}k_1 . \tag{17}$$

Since $\varepsilon(k_1) = \bar\varepsilon$ for $|k_1| < k_1^0$, the final potential $\Phi = \Phi_{\text{high}} + \Phi_{\text{low}}$ is obtained as

$$\Phi_0 = -\frac{e\beta^2\gamma^2}{8\pi^3} \int_{-\infty}^{\infty} F_0(\varepsilon, \varphi_0) \exp\{i k_1 x_{1(0)}\} \cdot C_0(k_1, x_2, x_3)\, \mathrm{d}k_1, \tag{18 a}$$

$$\Phi_e = -\frac{e\beta^2\gamma^2}{8\pi^3} \int_{-\infty}^{\infty} F_e(\varepsilon, \varphi_e) \exp\{i k_1 x_{1(e)}\} \cdot C_e(k_1, x_2, x_3)\, \mathrm{d}k_1 . \tag{18 b}$$

The angular parts of these integrals have the forms

$$F_0(\varepsilon, \varphi_0) = \frac{a_0^2 \sin^2\varphi_0}{D_0}, \qquad F_e(\varepsilon, \varphi_e) = \left[\Lambda^2 - m^2\cos^2\Omega_0 - \frac{\Lambda^2 a_e^2 \sin^2\varphi_e}{D_e} \right] \tag{19 a, b}$$

where

$$D_0 = p_{+(0)}\, p_{-(0)} + a_0\cos\varphi_0 (p_{+(0)} - p_{-(0)}) - a_0^2\cos^2\varphi_0 = \frac{\gamma^2}{\sin^2\Omega\cos^2\Omega_0} - (a_0\cos\varphi_0 - m^2\gamma\cot\Omega)^2, \tag{20 a}$$

$$D_e = p_{+(e)}\, p_{-(e)} + a_e\cos\varphi_e (p_{+(e)} - p_{-(e)}) - a_e^2\cos^2\varphi_e = \frac{\gamma^2}{\sin^2\Omega\cos^2\Omega_0} - \left(a_e\cos\varphi_e - \frac{\varepsilon_2}{\varepsilon_{22}}\gamma\cot\Omega \right)^2 . \tag{20 b}$$

Thus we see that the distortion integrals (3 c), which look so different from the isotropic integrals (3 a, b), combine with them in a natural way to give rise to azimuthal asymmetry in the potential distribution.

If we now assume the simplified dispersion curve discussed in I ($\varepsilon = 1$ for $|k_1| > k_1^0$), the angular factor $F(\varepsilon, \varphi)$ comes out of the integrand and the remaining integration of (18) reduces to the radial part already evaluated in I. The resulting potential is finite and oscillatory near the surface of the cone and is given by

$$\Phi = \sqrt{\frac{\pi}{2}}\, Q\, F(\bar\varepsilon, \varphi) \left\{ \frac{C(\sqrt{z}) \mp S(\sqrt{z})}{\sqrt{z}} \right\}. \tag{21}$$

In this expression, $z = \eta k_1^0$ where η denotes the radial distance of the field point from the surface of the cone, and Q is the potential at the site of the non-dispersive field cone. The upper and lower signs in (21) refer to field points inside and outside the field cone.

4. Field in a Doubly Anisotropic Medium

Using a theorem established elsewhere [9], the field in a doubly anisotropic medium can be determined without a fresh calculation simply by making certain substitutions in the expressions already derived for a pure dielectric. Although the substitution scheme works for motion of the charge in an arbitrary direction in a uniaxial or even a biaxial medium, the results become cumbersome in the general case. We therefore confine our discussions to the special case of motion in a principal plane of a medium satisfying the condition, $\mu_2 \varepsilon_3 = \mu_3 \varepsilon_2$. Under the more restrictive conditions $\varepsilon_2 = \varepsilon_3$ and $\mu_2 = \mu_3$, rotational symmetry round the optic axis renders any plane equivalent to a principal plane.

Letting (22)

$$\gamma k_1 \to k_1, \quad x_1 \to \gamma x_1, \quad \gamma^2 \beta^2 k_1^2 \varepsilon_{ij} \to \varepsilon_{ij}$$

we make the following substitutions in the expressions of the previous section for the potential Φ:

$$x_1 = \cos \Omega \, \bar{x}_1 + \sin \Omega \, \bar{x}_2, \quad x_2 = -\sin \Omega \, \bar{x}_1 + \cos \Omega \, \bar{x}_2,$$
$$x_3 = \bar{x}_3, \quad \bar{x}_i \to (\lambda_i / |\lambda|)^{1/2} \bar{x}_i, \quad k_1 \to (|\lambda|/\lambda_{11})^{1/2} k_1,$$
$$\varepsilon_i \to \varepsilon_i \lambda_i, \quad \cos \Omega \to (\lambda_1/\lambda_{11})^{1/2} \cos \Omega, \tag{23}$$
$$\sin \Omega \to (\lambda_2/\lambda_{11})^{1/2} \sin \Omega .$$

On reversing the step (22) and multiplying the final result by $(\lambda_{11}/|\lambda|)^{1/2}$ we obtain the desired expressions for Φ in a doubly anisotropic medium. In the above expressions, λ is the reciprocal permeability and $x_i, \bar{x}_i$ are the coordinates of the field point in the rest frame of the charge and in the principal axes frame respectively, with x_1-axis along the direction of motion of the charge. Apart from the overall multiplication factor, the potential Φ is again found to be given by Eqs. (6), (18) and (19), the symbols now having the altered meanings:

$$\varrho_0^2 = x_2^2 + \frac{\mu_{22}}{\mu_3} x_3^2, \quad x_{1(0)} = x_1 - \frac{\gamma \mu_{12}}{\mu_{22}} x_2,$$
$$a_0^2 = \frac{\gamma^2 (\beta^2 \varepsilon_3 \mu_{22} - 1) \mu_1 \mu_2}{\mu_{22}^2}, \quad \varrho_e^2 = x_2^2 + \frac{\varepsilon_{22}}{\varepsilon_3} x_3^2,$$
$$x_{1(e)} = x_1 - \frac{\gamma \varepsilon_{12}}{\varepsilon_{22}} x_2, \quad a_e^2 = \frac{\gamma^2 (\beta^2 \mu_3 \varepsilon_{22} - 1) \varepsilon_1 \varepsilon_2}{\varepsilon^2_{22}},$$
$$\Lambda^2 = \frac{\varepsilon_{22}}{\varepsilon_2} \cdot \frac{\mu_2}{\mu_{22}}, \quad m^2 = \frac{\mu_2}{\mu_{22}}, \tag{24}$$
$$\cos^2 \Omega_0 = 1/\beta^2 \varepsilon_2 \mu_3 = 1/\beta^2 \mu_2 \varepsilon_3, \quad p_{\pm(0)} = p_{\pm},$$
$$p_{\pm(e)} = p_{\pm} \pm \frac{\gamma (\varepsilon_2 \mu_1 - \mu_2 \varepsilon_1) \sin \Omega \cos \Omega}{\varepsilon_{22} \mu_{22}} .$$

The above expressions, as expected, are symmetric in ε and μ, and the equations of the o- and e-cones are now connected by the transformation $\varepsilon \longleftrightarrow \mu$. Since both the cones are elliptic in cross-section, the terms "ordinary" and "extraordinary" do not seem to possess any special significance in the doubly anisotropic case. While the o-cone is circular in a pure dielectric, the e-cone becomes circular in a pure ferrite and is commonly referred to as the o-cone in the literature on ferrites [10]. The overall multiplication factor of $(\mu_3 \mu_{22})^{1/2}$ signifies an enhanced output of radiation in a magnetic medium [11] and the consequent superiority of ferrites over dielectrics in the Cherenkov-generation of microwaves [12].

When the charge moves perpendicular to the optic axis, the angular parts simplify to

$$F_0(\varepsilon, \mu, \varphi_0) = \left[\frac{(\beta^2 \varepsilon_3 \mu_1 - 1) \sin^2 \varphi_0}{\beta^2 \varepsilon_3 \mu_1 \sin^2 \varphi_0 + \cos^2 \varphi_0} \right], \tag{25 a}$$

$$F_e(\varepsilon, \mu, \varphi_e) = \frac{1}{\beta^2 \varepsilon_3 \mu_1} \left[\frac{(\beta^2 \mu_3 \varepsilon_1 - 1) \cos^2 \varphi_e}{\beta^2 \mu_3 \varepsilon_1 \sin^2 \varphi_e + \cos^2 \varphi_e} \right]. \tag{25 b}$$

Thus, the field exhibits high azimuthal asymmetry in this case, the potential of the o-cone vanishing on the plane contained by the optic axis and the line of motion of the charge, and having a maximum value on the plane perpendicular to the optic axis. The situation is reversed for the e-cone. The radiation cones, being reciprocal to the field cones, must exhibit the same azimuthal asymmetry, and the rings of light emitted must have corresponding maxima and minima. These conclusions agree with Pafamov's findings on the energy radiated in a ferrite [10]. When the direction of motion of the charge is inclined to the optic axis at an arbitrary angle, the axes of the cones no longer coincide with the direction of motion and the azimuthal distribution is given by the more complicated expressions (19). For motion along the optic axis, the o-cone is totally extinguished and the e-cone becomes circular with a uniform azimuthal distribution of potential.

[1] V. L. Ginzburg, 'Propagation of Electromagnetic Waves in Plasma', Gordon and Breach, Science Publishers Inc., New York 1961.
[2] A. A. Kolomenskii, Dokl. Akad. Nauk. SSSR **106** (6), 982 [1956].
[3] J. F. McKenzie, Phil. Trans. Roy. Soc. London A **255**, 585 [1963].
[4] L. Marshall, Astrophys. J. **124**, 469 [1956].
[5] V. I. Veksler, CERN Symposium, Geneva 1956, Vol. 1, p. 80.
[6] S. Datta Majumdar and G. P. Sastry, Z. Naturforsch. **28 a**, 907 [1973].
[7] S. Datta Majumdar and R. Pal, Proc. Roy. Soc. London A **316**, 525 [1970].
[8] J. V. Jelley, 'Čerenkov Radiation and its Applications', Pergamon Press, London 1958.
[9] S. Datta Majumdar, Ann. Phys. New York **76**, (No. 2) 428 [1973].
[10] V. E. Pafamov, Zh. Eksp. Teor. Fiz. **30**, 761 (Soviet Phys. JETP. **3**, 597 [1956]).
[11] A. G. Sitenko, Dokl. Akad. Nauk. SSSR **98**, 377 [1954].
[12] H. Lashinsky, J. App. Phys. **27** (6), 631 [1956].